6 Topic: Proving geometric theorems by using complex numbers

The goal of this project is to practice the geometric properties of complex numbers. In all three theorems below you need to introduce complex numbers as points (or vectors) corresponding to certain vertices, and then use the geometric properties of addition and multiplication of complex numbers to obtain a purely algebraic proof. Please for each theorem use the format that we used for theorems of Euclidian geometry; that is, in addition to each statement I would like you to clearly specify what is "Given:" and what is expected "To be proven:".

♢ **6.1.**

Theorem (Bottema's theorem)**.** *In any given triangle ABC construct squares on any two adjacent sides, e.g., AC and BC. Then the midpoint of the segment that connects the vertices of the squares opposite to the common vertex C is independent of the position of C.*

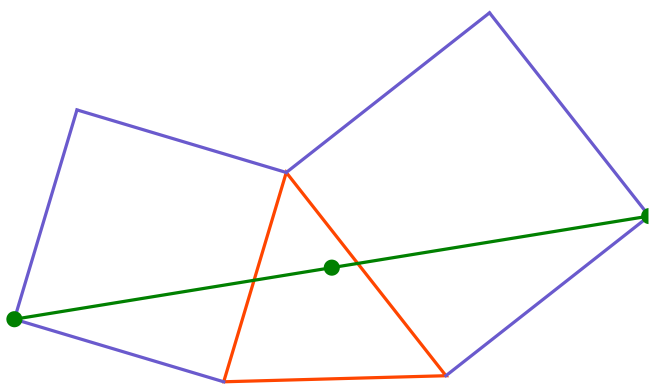

Figure 1: Illustration of Bottema's theorem.

Hints: Start with assuming that the vertices of the triangle correspond to the complex numbers $-1, +1$ and z. Next, find the complex numbers that correspond to the end points of the segment in question. At this step recall that multiplication by i corresponds to the counter-clockwise rotation by $\pi/2$ and multiplication by $-$i corresponds to the clockwise rotation by $\pi/2$. Finally recall the formula for the middle of the given segment and show that it does not depend on the point z.

♢ **6.2.**

Theorem (van Aubel's theorem)**.** *In any given quadrilateral $ABCD$ the segments connecting the centers of opposite squares built on the sides of $ABCD$ are of equal length and orthogonal to each other.*

Hints: Start with assigning to the vertices of the quadrilateral the complex numbers $2a, 2b, 2c, 2d$ (there are factors 2 here since you will need coordinates of the middles of each side, which look better if you start with factors 2) and then determine the coordinates (the corresponding complex numbers) of the centers of each of four squares. Finally compute two vectors (two complex numbers) that correspond to each of the segment in question, and use one complex multiplication to finish your proof.

♢ **6.3.**

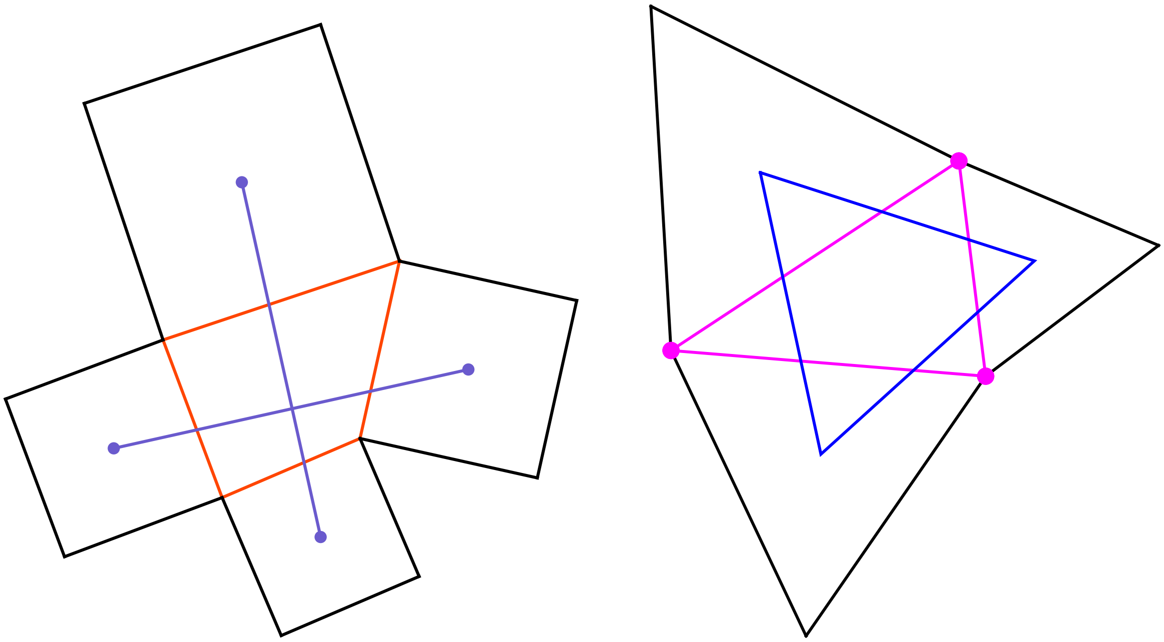

Figure 2: Illustrations of van Aubel's (left) and Napoleon's (right) theorems.

Theorem (Napoleon's theorem)**.** *If equilateral triangles are constructed on the sides of any triangle, the segments connecting the centers of those equilateral triangles themselves form an equilateral triangle.*

Hint: As in the previous problem, use $2a, 2b, 2c$ to denote the complex numbers that correspond to the vertices of the given triangle. One (small) complication is that now you will need to be careful with the coordinates of the centers of the equilateral triangles: recall that it is at the point of intersection of medians and divides all the medians in the ratio $2:1$.